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The Plasma Focus-Scaling Properties to Scaling Laws

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Abstract

Recent extensive and systematic numerical experiments have uncovered new insights into plasma focus devices including the following: (i) a plasma current limitation effect, as device static inductance $L_0$ is reduced towards very small values; (ii) scaling laws of neutron yield $Y_n$ and soft x-ray yield $Y_{sxr}$ as functions of storage energies $E_0$ and currents $I$; (iii) a global scaling law for neutron yield $Y_n$ as a function of storage energy $E_0$ combining experimental and numerical data showing that scaling deterioration has probably been interpreted as neutron ‘saturation’; and (iv) a fundamental cause of neutron ‘saturation’. An important scaling property is that the plasma condition is the same whether the plasma focus is a small sub-kilojoule machine or a large one with thousands of kilojoules of stored energy; and the related constancy of the dynamic resistance. This scaling property turns out to be the cause of ‘scaling deterioration’ of yields. The understanding of this situation points to a new class of plasma focus devices to overcome the ‘saturation’ of $I$ and yields. Plasma focus technology has to move to ultra-high voltage technology and take advantage of circuit manipulation techniques in order to move into a new era of high performance. This paper will examine fundamental scaling properties of the plasma focus including speeds and dynamic resistance, temperatures, dimensions and times, these being computed in the model which thus is a good source of reference for diagnostics. More importantly, we link up these basic scaling characteristics with the crucial ideas of the inherent yield scaling deterioration, thus providing a clear understanding of its overall performance characteristics, paving the way for future exploitation. The paper also takes a peek at the latest development of modeling the instability phase using anomalous resistance terms, resulting in quantitative experimental data of the instability phase of the plasma focus.

Keywords: Plasma Focus, Nuclear Fusion, Plasma Focus Scaling, Plasma Focus Properties, Neutron Saturation

1. Introduction

The plasma focus is one of the smaller scale devices which complements the international efforts to build a nuclear fusion reactor [1,2]. It is an important device for the generation of intense multi-radiation including x-rays, particle beams and fusion neutrons. The physics underlying the mechanisms for the generation of these radiations in the plasma focus is still not completely known although there have been intensive investigations for the past five decades. Experimental and theoretical work on the focus has reached quite high levels. For example, detailed simulation work on the plasma focus had been carried out since 1971 [3] and a large range of devices has been constructed from sub-kJ focus [4] to greater than 1 MJ large focus. Advanced experiments have been carried out on the dynamics, radiation, instabilities and non-linear phenomena [5]. Yet despite all these intensive studies, very little regarding scaling appears to be documented with the exception of the scaling law for neutron yield. Other more recent work has thrown much needed light on other aspects of scaling such as how the dimensions of the dense focused plasma (the focus pinch) and the pinch lifetime scale with apparatus dimensions, the dominating dimension being the anode radius [5-7].
2. Neutron scaling with energy

Historically the most appealing quantity for use as the base for scaling is the stored energy used to drive the focus. Using the highest voltage technologically convenient all one needs to do to scale up energy \( (E_0=0.5C_0V_0^2) \) is to put more capacitors in parallel, thus increasing the capacitance \( C_0 \) of the energy bank and incidentally also decreasing the static inductance \( L_0 \) of the bank to some extent. Along these lines, early work has shown that \( Y_n \sim E_0^2 \) \([5,9]\). Under ideal conditions (minimized inductance \( L_0 \) and when the system is dominated by the generator impedance) the capacitor current \( I \) may have the relationship \( I \sim E_0^{0.5} \), then it quickly follows that \( Y_n \sim I^4 \). This very simplistic view has led to the hold-up of the progress of large plasma focus devices. It was found that when the capacitor bank reached storage energies of only several hundred kJ the neutron yield no longer increased; the so-called neutron saturation effect \([5]\). It has been shown recently that whilst the discharge circuit is indeed dominated by the generator impedance at low energies (i.e. low \( C_0 \)) so that indeed \( I \sim E_0^{0.5} \); at a certain point when the \( C_0 \) (i.e. \( E_0 \)) gets sufficiently big, the generator impedance has dropped to such low values as to reach the value of the load impedance that the generator is driving. As \( E_0 \) is increased even further and further, the generator impedance eventually becomes negligible when compared to the load impedance which remains relatively constant, hardly affected by the decreasing generator impedance \([10,11]\). Eventually at very large \( E_0 \), the constant load impedance completely dominates and the circuit current reaches an asymptotic value and hardly increases for any further increase in \( E_0 \) at those already very large values. At this point which would be beyond the high tens of MJ for the plasma focus, the capacitor current may be considered to have saturated, leading to neutron saturation. What is observed at hundreds of kJ and which has been termed as neutron saturation is based on very limited data. When more data from more experiments are put together with data from rigorous systematic numerical experiments, then the global picture shows the scaling deterioration very clearly (see Figure 1) \([11]\). We will come back to this central problem again in the Section 9 of this paper.

![Figure 1. The global scaling law, combining experimental and numerical data. The global data illustrates \( Y_n \) scaling observed in numerical experiments from 0.4 kJ to 25 MJ (solid line) using the Lee model code, compared to measurements compiled from publications (squares) of various machines from 0.4 kJ to 1 MJ.](image-url)
3. Scaling Properties of the Plasma Focus

3.1. Various Plasma Focus Devices

In Figure 2a is shown the UNU ICTP PFF 3 kJ device [12] mounted on a 1m by 1m by 0.5m trolley, which was wheeled around the ICTP for the 1991 and 1993 Plasma Physics Colleges. The single capacitor is seen in the picture mounted on the trolley. In contrast, Figure 2b shows the 300-times larger PF1000, the 1 MJ device at the ICDMP in Warsaw Poland [13]. Only the chamber and the cables connecting the plasma focus to the capacitors are shown. The capacitor bank with its 288 capacitors, switches and chargers are located in a separate hall.

![Figure 2a. 3 kJ UNU ICTP PFF](image1.png)

![Figure 2b. 1 MJ PF1000 plasma focus](image2.png)

In order to throw further light on aspects of scaling such as how the dimensions of the focused pinch and the pinch lifetime scale with apparatus dimensions we have compiled tables (see following two sections) from numerical experiments, involving small and large plasma focus devices with a view of finding the relationship among relevant scaling properties.

3.2. Scaling Properties: mainly axial phase and neutron yield

We show in Table 1 the characteristics of three plasma focus devices [12-16] computed using the Lee model code [14,16], fitted by comparing the computed current waveform to the measured current waveform. These computed characteristics are also in broad agreement with measured experimental values where available in the published literature [12-16].

<table>
<thead>
<tr>
<th></th>
<th>$E_0$</th>
<th>A</th>
<th>$z_0$</th>
<th>$V_0$</th>
<th>$P_0$</th>
<th>$I_{\text{peak}}$</th>
<th>$v_a$</th>
<th>ID</th>
<th>SF</th>
<th>$Y_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>kJ</td>
<td>cm</td>
<td>cm</td>
<td>kV</td>
<td>Torr</td>
<td>kA/cm/μs</td>
<td>cm/cm</td>
<td>kA/cm( kA/cm) torr$^{0.5}$</td>
<td>10$^8$</td>
<td></td>
</tr>
<tr>
<td>PF1000</td>
<td>486</td>
<td>11.6</td>
<td>60</td>
<td>27</td>
<td>4</td>
<td>1850</td>
<td>11</td>
<td>160</td>
<td>85</td>
<td>1100</td>
</tr>
<tr>
<td>UNU ICTP</td>
<td>2.7</td>
<td>1.0</td>
<td>15.5</td>
<td>14</td>
<td>3</td>
<td>164</td>
<td>9</td>
<td>173</td>
<td>100</td>
<td>0.20</td>
</tr>
<tr>
<td>PF-400J</td>
<td>0.4</td>
<td>0.6</td>
<td>1.7</td>
<td>28</td>
<td>7</td>
<td>126</td>
<td>9</td>
<td>210</td>
<td>82</td>
<td>0.01</td>
</tr>
</tbody>
</table>

In Table 1 we look at the PF1000 and study its properties at typical operation with device storage at 500 kJ level. We compare this big focus with two small devices at the kJ and sub-kJ level.
We note:
Voltage and pressure do not have any particular relationship to $E_0$.
Peak current $I_{\text{peak}}$ increases with $E_0$.
Anode radius ‘a’ increases with $E_0$.
Current per cm of anode radius (ID) $I_{\text{peak}}/a$ is in a narrow range 160 to 210 kA/cm.
SF (speed or drive factor) $(I_{\text{peak}}/a)/P_0^{0.5}$ is 82 to 100 (kA/cm) torr$^{-0.5}$ deuterium gas [6].
Peak axial speed $v_a$ is in the narrow range 9 to 11 cm/us.
Fusion neutron yield $Y_n$ ranges from $10^6$ for the smallest device to $10^{11}$ for the PF1000.

We stress that whereas the ID and SF are practically constant at around 180 kA/cm and 90 (kA/cm) per torr$^{-0.5}$ deuterium gas throughout the range of small to big devices, $Y_n$ changes over 5 orders of magnitude.

We emphasize that the data of Table 1 is generated from numerical experiments and most of the data has been confirmed by actual experimental measurements and observation.

3.3. Scaling Properties: mainly radial phase and focus pinch

Table 2.

<table>
<thead>
<tr>
<th></th>
<th>$c$= a/b</th>
<th>$T_{\text{pinch}}$</th>
<th>$v_p$</th>
<th>$r_{\text{min}}$</th>
<th>$z_{\text{max}}$</th>
<th>Pinch duration</th>
<th>$r_{\text{min}}/a$</th>
<th>$z_{\text{max}}/a$</th>
<th>Pinch duration/a</th>
</tr>
</thead>
<tbody>
<tr>
<td>CM</td>
<td>10$^4$K</td>
<td>cm/µs</td>
<td>cm</td>
<td>cm</td>
<td>ns</td>
<td>ns/cm</td>
<td>cm</td>
<td>cm</td>
<td>cm</td>
</tr>
<tr>
<td>PF1000</td>
<td>1.4</td>
<td>11.6</td>
<td>2</td>
<td>13</td>
<td>2.2</td>
<td>19</td>
<td>165</td>
<td>0.17</td>
<td>1.6</td>
</tr>
<tr>
<td>UNU ICTP</td>
<td>3.4</td>
<td>1.0</td>
<td>8</td>
<td>26</td>
<td>0.13</td>
<td>1.4</td>
<td>7.3</td>
<td>0.14</td>
<td>1.4</td>
</tr>
<tr>
<td>PF400J</td>
<td>2.6</td>
<td>0.6</td>
<td>6</td>
<td>23</td>
<td>0.09</td>
<td>0.8</td>
<td>5.2</td>
<td>0.14</td>
<td>1.4</td>
</tr>
</tbody>
</table>

Table 2 compares further the properties of the range of plasma focus devices. The properties compared in this table are mainly related to the radial phase. These are computed from numerical experiments and found to be in close agreement with laboratory measurements [12-16].

We note:
Pinch temperature $T_{\text{pinch}}$ is strongly correlated to the square of the radial pinch speed $v_p$.
$v_p$ itself is closely correlated to the value of $v_a$ and $c=b/a$; so that for a constant $v_a$, $v_p$ is almost proportional to the value of $c=b/a$ [14,16].
Dimensions and lifetime of the focus pinch scales as the anode radius ‘a’:
$r_{\text{min}}/a$ (almost constant at 0.14-0.17)
$z_{\text{max}}/a$ (almost constant at 1.5)
Pinch duration has a relatively narrow range of 8-14 ns/cm of anode radius.
Duration per unit anode radius is correlated to the inverse of $T_{\text{pinch}}$.

$T_{\text{pinch}}$ itself is a measure of the energy per unit mass. It is quite remarkable that this energy density at the focus pinch varies so little (factor of 4) over a range of device energy of more than 3 orders of magnitude (factor of 1000).

This practically constant pinch energy density (per unit mass) is related to the constancy of the axial speed moderated by the effect of the values of $c=b/a$ on the radial speed.
The constancy of \( r_{\text{min}}/a \) suggests that the devices also produce the same compression of ambient density to maximum pinch density; with the ratio (maximum pinch density)/(ambient density) being proportional to \((a/r_{\text{min}})^2\). So for two devices of different sizes starting with the same ambient fill density, the maximum pinch density would be the same.

### 3.4. Scaling Properties: Rules- of- Thumb

From the above discussions, we may put down as rule-of-thumb the following scaling relationships, subject to minor variations caused primarily by the variation in \( c=b/a \).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial phase energy density (per unit mass)</td>
<td>constant</td>
</tr>
<tr>
<td>Radial phase energy density (per unit mass)</td>
<td>constant</td>
</tr>
<tr>
<td>Pinch radius ratio</td>
<td>constant</td>
</tr>
<tr>
<td>Pinch length ratio</td>
<td>constant</td>
</tr>
<tr>
<td>Pinch duration per unit anode radius</td>
<td>constant</td>
</tr>
<tr>
<td>Dynamic resistance, axial phase</td>
<td>constant</td>
</tr>
</tbody>
</table>

Summarising, the dense hot plasma pinch of a small \( E_0 \) plasma focus and that of a big \( E_0 \) plasma focus have essentially the same energy density \([6-8]\), the same mass density. The big \( E_0 \) plasma focus has a bigger physical size and a bigger discharge current. The size of the plasma pinch scales proportionately to the current and to the anode radius, as does the duration of the plasma pinch. The bigger \( E_0 \), the bigger \( I_{\text{peak}} \), the bigger ‘a’ has to be, correspondingly the larger the plasma pinch radius and the longer the duration of the plasma pinch. The larger size and longer duration of the big \( E_0 \) plasma pinch are essentially the properties leading to the bigger neutron yield compared to the yield of the small \( E_0 \) plasma focus. We have also included that the axial phase dynamic resistance is a constant as a rule-of-thumb. This is related to the constant axial phase energy density but is listed here as it plays a predominant role in the physical mechanism of deterioration of yield scaling which will be discussed in greater detail in Section 9.2.

### 3.5. Dimensions and Lifetimes of the plasma focus in D and Ne

We may also summarise the dimensions and lifetimes for deuterium and neon plasma focus pinch as follows \([7,8]\):

<table>
<thead>
<tr>
<th></th>
<th>Deuterium</th>
<th>Neon (for SXR)</th>
</tr>
</thead>
<tbody>
<tr>
<td>minimum radius ( r_{\text{min}} )</td>
<td>0.15a</td>
<td>0.05a</td>
</tr>
<tr>
<td>max length (hollow anode)</td>
<td>Z 1.5a</td>
<td>1.6a</td>
</tr>
<tr>
<td>radial shock transit ( t_{\text{comp}} )</td>
<td>( 5 \times 10^{-6} )a</td>
<td>( 4 \times 10^{-6} )a</td>
</tr>
<tr>
<td>pinch lifetime ( t_p )</td>
<td>( 10^{16} )a</td>
<td>( 10^{15} )a</td>
</tr>
<tr>
<td>Speed factor SF</td>
<td>90</td>
<td></td>
</tr>
</tbody>
</table>

Where, for the times in sec, the value of anode radius, a, is in m. For the neon calculations radiative terms are included; and the stronger compression (smaller radius) is due to thermodynamic effects. The units of the speed factor SF are: \((\text{kA/cm})/(\text{torr}^{0.5})\)

The above description of the plasma focus combines data from numerical experiments using the Lee Model code, consistent with laboratory observations \([6-8,14,16]\). The next section describes briefly the code.
4. Introduction to the Lee model code

The Lee model code couples the electrical circuit with plasma focus dynamics, thermodynamics, and radiation, enabling a realistic simulation of all gross focus properties. The basic model, described in 1984 [17], was successfully used to assist several projects [12,18,19]. Radiation-coupled dynamics was included in the five-phase code, leading to numerical experiments on radiation cooling [20]. The vital role of a finite small disturbance speed discussed by Potter in a Z-pinch situation [21] was incorporated together with real gas thermodynamics and radiation-yield terms. This version of the code assisted other research projects [22-27] and was web published in 2000 [28] and 2005 [29]. Plasma self-absorption was included in 2007 [28], improving the SXR yield simulation. The code has been used extensively in several machines including UNU/ICTP PFF [12,14,16,20,22,24-27,30], NX2 [23,31,32], and NX1 [32,33] and has been adapted for the Filippov-type plasma focus DENA [34]. A recent development is the inclusion of the neutron yield $Y_n$ using a beam–target mechanism [10,11,35-39], incorporated in recent versions [14,16] of the code (versions later than RADPFV5.13), resulting in realistic $Y_n$ scaling with $I_{\text{pinch}}$ [10,11,35-38]. The versatility and utility of the model are demonstrated in its clear distinction of $I_{\text{pinch}}$ from $I_{\text{peak}}$ [36,37] and the recent uncovering of a plasma focus pinch current limitation effect [37-40], as static inductance is reduced towards small values. Extensive numerical experiments had been carried out systematically resulting in the uncovering of neutron [1,10,11,35,41-43,48] and SXR [41-48] scaling laws over a wider range of energies and currents than attempted before. The numerical experiments also gave insight into the nature and cause of ‘neutron saturation’ [1,10,11,42]. The description, theory, code, and a broad range of results of this “Universal Plasma Focus Laboratory Facility” are available for download from [14,16].

A brief description of the 5-phase model is given in the following.

4.1. The 5-phases

The five phases (a-e) are summarised [14,16, 35-49] as follows:

a. Axial Phase (see Figure 3 left part): Described by a snowplow model with an equation of motion which is coupled to a circuit equation. The equation of motion incorporates the axial phase model parameters: mass and current factors $f_m$ and $f_c$ [18,50]. The mass swept-up factor $f_m$ accounts for not only the porosity of the current sheet but also for the inclination of the moving current sheet-shock front structure, boundary layer effects, and all other unspecified effects
which have effects equivalent to increasing or reducing the amount of mass in the moving structure, during the axial phase. The current factor $f_c$ accounts for the fraction of current effectively flowing in the moving structure (due to all effects such as current shedding at or near the back-wall, and current sheet inclination). This defines the fraction of current effectively driving the structure, during the axial phase.

Figure 4. Schematic of radius versus time trajectories to illustrate the radial inward shock phase when $r_s$ moves radially inwards, the reflected shock (RS) phase when the reflected shock moves radially outwards, until it hits the incoming piston $r_p$ leading to the start of the pinch phase ($t_f$) and finally the expanded column phase.

b. Radial Inward Shock Phase (see Figure 3 right part, also Figure 4): Described by 4 coupled equations using an elongating slug model. The first equation computes the radial inward shock speed from the driving magnetic pressure. The second equation computes the axial elongation speed of the column. The third equation computes the speed of the current sheath, (magnetic piston), allowing the current sheath to separate from the shock front by applying an adiabatic approximation [21]. The fourth is the circuit equation. Thermodynamic effects due to ionization and excitation are incorporated into these equations, these effects being particularly important for gases other than hydrogen and deuterium. Temperature and number densities are computed during this phase using shock-jump equations. A communication delay between shock front and current sheath due to the finite small disturbance speed [14,16,21] is crucially implemented in this phase. The model parameters, radial phase mass swept-up and current factors $f_{mr}$ and $f_{cr}$ are incorporated in all three radial phases. The mass swept-up factor $f_{mr}$ accounts for all mechanisms which have effects equivalent to increasing or reducing the amount of mass in the moving slug, during the radial phase. The current factor $f_{cr}$ accounts for the fraction of current effectively flowing in the moving piston forming the back of the slug (due to all effects). This defines the fraction of current effectively driving the radial slug.

c. Radial Reflected Shock (RS) Phase (See Figure 4): When the shock front hits the axis, because the focus plasma is collisional, a reflected shock develops which moves radially outwards, whilst the radial current sheath piston continues to move inwards. Four coupled equations are also used to describe this phase, these being for the reflected shock moving radially outwards, the piston moving radially inwards, the elongation of the annular column and the circuit. The same model parameters $f_{mr}$ and $f_{cr}$ are used as in the previous radial phase. The plasma temperature behind the reflected shock undergoes a jump by a factor close to 2. Number densities are also computed using the reflected shock jump equations.
d. Slow Compression (Quiescent) or Pinch Phase (See Figure 4): When the out-going reflected shock hits the inward moving piston, the compression enters a radiative phase in which for gases such as neon, radiation emission may actually enhance the compression where we have included energy loss/gain terms from Joule heating and radiation losses into the piston equation of motion. Three coupled equations describe this phase; these being the piston radial motion equation, the pinch column elongation equation and the circuit equation, incorporating the same model parameters as in the previous two phases. The duration of this slow compression phase is set as the time of transit of small disturbances across the pinched plasma column. The computation of this phase is terminated at the end of this duration.

e. Expanded Column Phase: To simulate the current trace beyond this point we allow the column to suddenly attain the radius of the anode, and use the expanded column inductance for further integration. In this final phase the snow plow model is used, and two coupled equations are used similar to the axial phase above. This phase is not considered important as it occurs after the focus pinch.

We note [51] that in radial phases b, c and d, axial acceleration and ejection of mass caused by necking curvatures of the pinching current sheath result in time-dependent strongly center-peaked density distributions. Moreover the transition from phase d to phase e is observed in laboratory measurements to occur in an extremely short time with plasma/current disruptions resulting in localized regions of high densities and temperatures. These centre-peaking density effects and localized regions are not modeled in the code, which consequently computes only an average uniform density and an average uniform temperature which are considerably lower than measured peak density and temperature. However, because the four model parameters are obtained by fitting the computed total current waveform to the measured total current waveform, the model incorporates the energy and mass balances equivalent, at least in the gross sense, to all the processes which are not even specifically modeled. Hence the computed gross features such as speeds, trajectories and integrated soft x-ray yields have been extensively tested in numerical experiments for several machines and are found to be comparable with measured values.

5. Modeling as reference for diagnostics

5.1. Fitting the computed to the measured current waveforms

The Lee model code is configured [14,16,28,29, 35-49] to work as any plasma focus by inputting:
Bank parameters, $L_0$, $C_0$ and stray circuit resistance $r_0$;
Tube parameters $b$, $a$ and $z_0$ and
Operational parameters $V_0$ and $P_0$ and the fill gas.
The computed total current waveform is fitted to the measured waveform by varying model parameters $f_m$, $f_c$, $f_{mr}$ and $f_{cr}$ one by one, until the computed waveform agrees with the measured waveform.
First, the axial model factors $f_m$, $f_c$ are adjusted (fitted) until the features (1) computed rising slope of the total current trace and (2) the rounding off of the peak current as well as (3) the peak current itself are in reasonable (typically very good) fit with the measured total current trace (see Fig 5, measured trace fitted with computed trace).
Then we proceed to adjust (fit) the radial phase model factors $f_{mr}$ and $f_{cr}$ until features (4) the computed slope and (5) the depth of the dip agree with the measured. Note that the fitting of the computed trace with the measured current trace is done up to the end of the radial phase which is typically at the bottom of the current dip. Fitting of the computed and measured current traces
beyond this point is not done. If there is significant divergence of the computed with the measured trace beyond the end of the radial phase, this divergence is not considered important. In this case, after fitting the 5 features (1) to (5) above, the following fitted model parameters are obtained: $f_m=0.1, f_r=0.7, f_{mr}=0.12, f_{cr}=0.68$.

**Figure 5.** The 5-point fitting of computed current trace to the measured (or the reference) current trace. Point 1 is the current rise slope. Point 2 is the topping profile. Point 3 is the peak value of the current. Point 4 is the slope of the current dip. Point 5 is the bottom of the current dip. Fitting is done up to point 5 only. Further agreement or divergence of the computed trace with/from the measured trace is only incidental and not considered to be important.

**5.2. Philosophy for current fitting**

From experience it is known that the current trace of the focus is one of the best indicators of gross performance. The axial and radial phase dynamics and the crucial energy transfer into the focus pinch are among the important information that is quickly apparent from the current trace [14,16].

The exact time profile of the total current trace is governed by the bank parameters, by the focus tube geometry and the operational parameters. It also depends on the fraction of mass swept-up and the fraction of sheath current and the variation of these fractions through the axial and radial phases. These parameters determine the axial and radial dynamics, specifically the axial and radial speeds which in turn affect the profile and magnitudes of the discharge current. There are many underlying mechanisms in the axial phase such as shock front and current sheet structure, porosity and inclination, boundary layer effects and current shunting and fragmenting which are not simply modeled; likewise in the radial phase mechanisms such as current sheet curvatures and necking leading to axial acceleration and ejection of mass, and plasma/current disruptions. These effects may give rise to localized regions of high density and temperatures. The detailed profile of the discharge current is influenced by these effects and during the pinch phase also reflects the Joule heating and radiative yields. At the end of the pinch phase the total current profile also reflects the sudden transition of the current flow from a constricted pinch to a large column flow. Thus the discharge current powers all dynamic, electrodynamic, thermodynamic and radiation processes in the various phases of the plasma focus. Conversely all the dynamic, electrodynamic, thermodynamic and radiation processes in the various phases of the plasma focus affect the discharge current. It is then no exaggeration to say that the discharge current
waveform contains information on all the dynamic, electrodynamic, thermodynamic and radiation processes that occur in the various phases of the plasma focus. This explains the importance attached to matching the computed total current trace to the measured total current trace in the procedure adopted by the Lee model code. Once matched, the fitted model parameters assure that the computation proceeds with all physical mechanisms accounted for, at least in the gross energy and mass balance sense.

5.3. Diagnostics—Time histories of dynamics, energies and plasma properties computed from the measured total current waveform by the code

During every adjustment of each of the model parameters the code goes through the whole cycle of computation. In the last adjustment, when the computed total current trace is judged to be reasonably well fitted in all 5 waveform features, computed time histories are presented, in Fig 6a-6o as an example, as follows: for the NX2 operated at 11 kV, 2.6 Torr neon [14,16,31].

![Figure 6a. Fitted computed \( I_{total} \)](image)

![Figure 6b. Computed \( I_{total} \) & \( I_{plasma} \)](image)

![Figure 6c. Tube voltage](image)

![Figure 6d. Axial trajectory and speed](image)

![Figure 6e. Radial trajectories](image)

![Figure 6f. Length of elongating structure](image)
Figure 6g. Speeds in radial phases

Figure 6h. Tube inductance-axial & radial phases

Figure 6i. Total inductive energy

Figure 6j. Piston work and DR energy; both traces overlap

Figure 6k. DR axial and radial phases

Figure 6l. Peak & averaged uniform \( n_i \)

Figure 6m. Peak & averaged uniform \( n_e \)

Figure 6n. Peak and averaged uniform \( T \)

Figure 6o. Neon Soft x-ray power

5.4. Comments on computed quantities

The computed total current trace typically agrees very well with the measured because of the fitting. The end of the radial phase is indicated in Fig. 6a. Plasma currents are rarely measured. We had done a comparison of the computed plasma current with measured plasma current for the Stuttgart PF78 which shows good agreement of our computed to the measured plasma current [36]. The computed plasma current in this case of the NX2 is shown in Fig 6b. The computed
tube voltage is difficult to compare with measured tube voltages in terms of peak values, typically because of poor response time of voltage dividers. However the computed waveform shape in Fig 6c. is general as expected. The computed axial trajectory and speed, agree with experimental obtained time histories. Moreover, the behaviour with pressure, running the code again at different pressures, agrees well with experimental results. The radial trajectories and speeds are difficult to measure. The computed trajectories Fig 6e agree with the scant experimental data available. The length of the radial structure is shown in Fig 6f. Computed speeds radial shock front and piston speeds and speed of the elongation of the structure are shown in Fig 6g. The computed inductance (Fig 6h) shows a steady increase of inductance in the axial phase, followed by a sharp increase (rising by more than a factor of 2 in a radial phase time interval about 1/10 the duration of the axial phase for the NX2).

The inductive energy \( (0.5LI^2) \) peaks at 70% of initial stored energy, and then drops to 30% during the radial phase, as the sharp drop of current more than offsets the effect of sharply increased inductance (Fig 6i). In Fig 6j is shown the work done by the magnetic piston, computed using force integrated over distance method. Also shown is the work dissipated by the dynamic resistance, computed using dynamic resistance power integrated over time. We see that the two quantities and profiles agree exactly. This validates the concept of half \( L \dot{I} \) as a dynamic resistance, \( DR \) (see section 9.2). The piston work deposited in the plasma increases steadily to some 12% at the end of the axial phase and then rises sharply to just below 30% in the radial phase. Dynamic resistance (DR) is shown in Fig 6k. The values of the DR in the axial phase, together with the bank surge impedance, are the quantities that determine \( I_{\text{peak}} \). The ion number density has a maximum value derived from shock-jump considerations, and an averaged uniform value derived from overall energy and mass balance considerations. The time profiles of these are shown in the Fig 6l. The electron number density (Fig 6m) has similar profiles to the ion density profile, but is modified by the effective charge numbers due to ionization stages reached by the ions. Plasma temperature too has a maximum value and an averaged uniform value derived in the same manner; are shown in Fig 6n. Computed neon soft x-ray power profile is shown in Fig 6o. The area of the curve is the soft x-ray yield in Joule. Pinch dimensions and lifetime may be estimated from Figs 6e and 6f. The model also computes the neutron yield, for operation in deuterium, using a phenomenological beam-target mechanism [14,16,35-37]. The model does not compute a time history of the neutron emission, only a yield number \( Y_n \).

Thus as is demonstrated above, the model code when properly fitted is able to realistically model any plasma focus and act as a guide to diagnostics of plasma dynamics, trajectories, energy distribution and gross plasma properties.

6. Insights from Modelling

Moreover, using such simulation, series of experiments have been systematically carried out to look for behaviour patterns of the plasma focus. Insights uncovered by the series of experiments include: (i) pinch current limitation effect as static inductance is reduced; (ii) neutron and SXR scaling laws; (iii) a global scaling law for neutrons versus storage energy combining experimental and numerical experimental data; and (iv) insight into the nature and a fundamental cause of neutron saturation. These are significant achievements accomplished within a period of twenty months of intensive numerical experimentation.

6.1. Insight 1-Pinch Current Limitation Effect as Static Inductance is Reduced Towards Zero

In a recent paper [13] there was expectation that the large MJ plasma focus PF1000 in Warsaw could increase its discharge current, and its pinch current, and consequently neutron yield by a
reduction of its external or static inductance $L_0$. To investigate this point, experiments were carried out using the Lee Model code. Unexpectedly, the results indicated that whilst $I_{\text{peak}}$ indeed progressively increased with reduction in $L_0$, no improvement may be achieved due to a pinch current limitation effect [37,38]. Given a fixed $C_0$ powering a plasma focus, there exists an optimum $L_0$ for maximum $I_{\text{pinch}}$. Reducing $L_0$ further will increase neither $I_{\text{pinch}}$ nor $Y_n$. The numerical experiments leading to this unexpected result is described below.

A measured current trace of the PF1000 with $C_0 = 1332$ μF, operated at 27 kV, 3.5 torr deuterium, has been published [13], with cathode/anode radii $b = 16$ cm, $a = 11.55$ cm and anode length $z_0 = 60$ cm. In the numerical experiments we fitted external (or static) inductance $L_0 = 33.5$ nH and stray resistance $r_0 = 6.1$ mΩ (damping factor $RESF = r_0/(L_0/C_0)^{0.5} = 1.22$). The fitted model parameters are: $f_m = 0.13$, $f_c = 0.7$, $f_{mr} = 0.35$ and $f_{cr} = 0.65$. The computed current trace [14,35-39] agrees very well with the measured trace through all the phases, axial and radial, right down to the bottom of the current dip indicating the end of the pinch phase as shown in Fig.7.

We carried out numerical experiments for PF1000 using the machine and model parameters determined from Figure 7. Operating the PF1000 at 35 kV and 3.5 Torr, we varied the anode radius $a$ with corresponding adjustment to $b$ to maintain a constant $c=b/a=1.39$ and in order to keep the peak axial speed at 10 cm/μs. The anode length $z_0$ was also adjusted to maximize $I_{\text{pinch}}$ as $L_0$ was decreased from 100 nH progressively to 5 nH.

As expected, $I_{\text{peak}}$ increased progressively from 1.66 to 4.4 MA. As $L_0$ was reduced from 100 to 35 nH, $I_{\text{pinch}}$ also increased, from 0.96 to 1.05 MA. However, then unexpectedly, on further reduction from 35 to 5 nH, $I_{\text{pinch}}$ stopped increasing, instead decreasing slightly to 1.03 MA at 20 nH, to 1.0 MA at 10 nH, and to 0.97 MA at 5 nH. $Y_n$ also had a maximum value of 3.2x10$^{11}$ at 35 nH.

6.2. explaining the effect

To explain this unexpected result, we examine the energy distribution in the system at the end of the axial phase (see Fig 7) just before the current drops from peak value $I_{\text{peak}}$ and then again near the bottom of the almost linear drop to the pinch phase indicated by the arrow pointing to ‘end of radial phase’. The energy equation describing this current drop is written as follows:

$$0.5I_{\text{peak}}^2(L_0 + L_{afc}^2) = 0.5I_{\text{pinch}}^2(L_0/f_c^2 + L_a + L_p) + \delta_{\text{cap}} + \delta_{\text{plasma}},$$

(1)
where \( L_a \) is the inductance of the tube at full axial length \( z_0 \), \( \delta_{\text{plasma}} \) is the energy imparted to the plasma as the current sheet moves to the pinch position and is the integral of \( 0.5(dL/dt)I^2 \). We approximate this as \( 0.5L_pI_{\text{pinch}}^2 \) which is an underestimate for this case. \( \delta_{\text{cap}} \) is the energy flow into or out of the capacitor during this period of current drop. If the duration of the radial phase is short compared to the capacitor time constant, the capacitor is effectively decoupled and \( \delta_{\text{cap}} \) may be put as zero. From this consideration we obtain

\[
I_{\text{pinch}}^2 = I_{\text{peak}}^2((L_0 + 0.5L_a)/(2L_0 + L_a + 2L_p)),
\]

where we have taken \( f_c = 0.7 \) and approximated \( f_c^2 \) as 0.5.

Generally, as \( L_0 \) is reduced, \( I_{\text{peak}} \) increases; \( a \) is necessarily increased leading [9] to a longer pinch length \( z_p \), hence a bigger \( L_p \). Lowering \( L_0 \) also results in a shorter rise time, hence a necessary decrease in \( z_0 \), reducing \( L_a \). Thus, from Eq. (2), lowering \( L_0 \) decreases the fraction \( I_{\text{pinch}} / I_{\text{peak}} \).

Secondly, this situation is compounded by another mechanism. As \( L_0 \) is reduced, the \( L-C \) interaction time of the capacitor bank reduces while the duration of the current drop increases (see Fig 6-8, discussed in the next section) due to an increasing \( a \). This means that as \( L_0 \) is reduced, the capacitor bank is more and more coupled to the inductive energy transfer processes with the accompanying induced large voltages that arise from the radial compression. Looking again at the derivation of Eq. (2) from Eq. (1) a nonzero \( \delta_{\text{cap}} \), in this case, of positive value, will act to decrease \( I_{\text{pinch}} \) further. The lower the \( L_0 \) the more pronounced is this effect.

Summarizing this discussion, the pinch current limitation is not a simple effect, but is a combination of the two complex effects described above, namely, the interplay of the various inductances involved in the plasma focus processes abetted by the increasing coupling of \( C_o \) to the inductive energetic processes, as \( L_0 \) is reduced.

### 6.3. Optimum \( L_0 \) for maximum pinch current and neutron yield

From the pinch current limitation effect, it is clear that given a fixed \( C_o \) powering a plasma focus, there exists an optimum \( L_0 \) for maximum \( I_{\text{pinch}} \). Reducing \( L_0 \) further will increase neither \( I_{\text{pinch}} \) nor \( Y_n \). The results of the numerical experiments carried out are presented in Figure 8 and Table 4.

With large \( L_0 = 100 \text{ nH} \) it is seen (Figure 8) that the rising current profile is flattened from what its waveform would be if unloaded; and peaks at around 12 s (before its unloaded rise time, not shown, of 18 s) as the current sheet goes into the radial phase. The current drop, less than 25% of peak value, is sharp compared with the current rise profile. At \( L_0 = 30 \text{ nH} \) the rising current profile is less flattened, reaching a flat top at around 5 s, staying practically flat for some 2 s before the radial phase current drop to 50% of its peak value in a time which is still short compared with the rise time. With \( L_0 = 5 \text{ nH} \), the rise time is now very short, there is hardly any flat top; as soon as the peak is reached, the current waveform droops significantly. There is a small kink on the current waveform of both the \( L_0 = 5 \text{ nH}, z_0 = 20 \text{ cm} \) and the \( L_0 = 5 \text{ nH}, z_0 = 40 \text{ cm} \). This kink corresponds to the start of the radial phase which, because of the large anode radius, starts with a relatively low radial speed, causing a momentary reduction in dynamic loading. Looking at the three types of traces it is seen that for \( L_0 = 100 \text{ nH} \) to 30 nH, there is a wide range of \( z_0 \) that may be chosen so that the radial phase may start at peak or near peak current, although the longer values of \( z_0 \) tend to give better energy transfers into the radial phase.
The optimized situation for each value of \( L_0 \) is shown in Table 4. The table shows that as \( L_0 \) is reduced, \( I_{\text{peak}} \) rises with each reduction in \( L_0 \) with no sign of any limitation. However, \( I_{\text{pinch}} \) reaches a broad maximum of 1.05MA around 40–30 nH. Neutron yield \( Y_n \) also shows a similar broad maximum peaking at \( 3.2 \times 10^{11} \) neutrons. Figure 9 shows a graphical representation of this \( I_{\text{pinch}} \) limitation effect. The curve going up to 4MA at low \( L_0 \) is the \( I_{\text{peak}} \) curve. Thus \( I_{\text{peak}} \) shows no sign of limitation as \( L_0 \) is progressively reduced. However \( I_{\text{pinch}} \) reaches a broad maximum. From Fig 9 there is a stark and important message. One must distinguish clearly between \( I_{\text{peak}} \) and \( I_{\text{pinch}} \). In general one cannot take \( I_{\text{peak}} \) to be representative of \( I_{\text{pinch}} \).

Table 4. Currents and ratio of currents as \( L_0 \) is reduced-PF1000 at 35kV, 3.5 Torr Deuterium

<table>
<thead>
<tr>
<th>( L_0 ) (nH)</th>
<th>( b ) (cm)</th>
<th>( a ) (cm)</th>
<th>( z_0 ) (cm)</th>
<th>( I_{\text{peak}} ) (MA)</th>
<th>( I_{\text{pinch}} ) (M)</th>
<th>( Y_n ) ( (10^{11}) )</th>
<th>( I_{\text{pinch}} / I_{\text{peak}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>15.0</td>
<td>10.8</td>
<td>80</td>
<td>1.66</td>
<td>0.96</td>
<td>2.44</td>
<td>0.58</td>
</tr>
<tr>
<td>80</td>
<td>16.0</td>
<td>11.6</td>
<td>80</td>
<td>1.81</td>
<td>1.00</td>
<td>2.71</td>
<td>0.55</td>
</tr>
<tr>
<td>60</td>
<td>18.0</td>
<td>13.0</td>
<td>70</td>
<td>2.02</td>
<td>1.03</td>
<td>3.01</td>
<td>0.51</td>
</tr>
<tr>
<td>40</td>
<td>21.5</td>
<td>15.5</td>
<td>55</td>
<td>2.36</td>
<td>1.05</td>
<td>3.20</td>
<td>0.44</td>
</tr>
<tr>
<td>35</td>
<td>22.5</td>
<td>16.3</td>
<td>53</td>
<td>2.47</td>
<td>1.05</td>
<td>3.20</td>
<td>0.43</td>
</tr>
<tr>
<td>30</td>
<td>23.8</td>
<td>17.2</td>
<td>50</td>
<td>2.61</td>
<td>1.05</td>
<td>3.10</td>
<td>0.40</td>
</tr>
<tr>
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<td>28.0</td>
<td>21.1</td>
<td>32</td>
<td>3.13</td>
<td>1.03</td>
<td>3.00</td>
<td>0.33</td>
</tr>
<tr>
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<td>33.0</td>
<td>23.8</td>
<td>28</td>
<td>3.65</td>
<td>1.00</td>
<td>2.45</td>
<td>0.27</td>
</tr>
<tr>
<td>5</td>
<td>40.0</td>
<td>28.8</td>
<td>20</td>
<td>4.37</td>
<td>0.97</td>
<td>2.00</td>
<td>0.22</td>
</tr>
</tbody>
</table>

Figure 8. PF1000 current waveforms computed at 35kV, 3.5 Torr \( D_2 \) for a range of \( L_0 \) showing the changes in waveforms as \( L_0 \) varies.

Figure 9. Currents and current ratio (computed) as \( L_0 \) is reduced PF1000, 35 kV, 3.5 torr \( D_2 \).
We carried out several sets of experiments on the PF1000 for varying $L_0$, each set with a different damping factor. In every case, an optimum inductance was found around 30–60 nH with $I_{\text{pinch}}$ decreasing as $L_0$ was reduced below the optimum value. The results showed that for PF1000, reducing $L_0$ from its present 20–30 nH will increase neither the observed $I_{\text{pinch}}$ nor the neutron yield, because of the pinch limitation effect. Indeed, the $I_{\text{pinch}}$ decreases very slightly on further reduction to very small values. We would add that we have used a set of model parameters which in our experience is the most reasonable to be used in these numerical experiments. Variations of the model parameters could occur but we are confident that these variations are not likely to occur with such a pattern as to negate the pinch current limitation effect. Nevertheless these variations should be actively monitored and any patterns in the variations should be investigated.

7. Insight 2-Scaling Laws for Neutron

7.1. Computation of Neutron yield-describing the beam-target mechanism

The neutron yield is computed using a phenomenological beam-target neutron generating mechanism described recently by Gribkov et al [13] and adapted to yield the following equation. A beam of fast deuteron ions is produced by diode action in a thin layer close to the anode, with plasma disruptions generating the necessary high voltages. The beam interacts with the hot dense plasma of the focus pinch column to produce the fusion neutrons. The beam-target yield is derived [14,16, 35-39] as:

$$ Y_{\text{bt}} = C_n n_i I_{\text{pinch}}^2 z_p^2 (\ln (b/r_p)) \sigma / U^{0.5} $$

where $n_i$ is the ion density, $b$ is the cathode radius, $r_p$ is the radius of the plasma pinch with length $z_p$, $\sigma$ the cross-section of the D-D fusion reaction, n- branch [52] and $U$, the beam energy. $C_n$ is treated as a calibration constant combining various constants in the derivation process.

The D-D cross-section is sensitive to the beam energy in the range 15-150 kV; so it is necessary to use the appropriate range of beam energy to compute $\sigma$. The code computes induced voltages (due to current motion inductive effects) $V_{\text{max}}$ of the order of only 15-50 kV. However it is known, from experiments that the ion energy responsible for the beam-target neutrons is in the range 50-150 keV [5,13], and for smaller lower-voltage machines the relevant energy could be lower at 30-60 keV [27]. Thus in line with experimental observations the D-D cross section $\sigma$ is reasonably obtained by using $U = 3V_{\text{max}}$. This fit was tested by using $U$ equal to various multiples of $V_{\text{max}}$. A reasonably good fit of the computed neutron yields to the measured published neutron yields at energy levels from sub-kJ to near MJ was obtained when the multiple of 3 was used; with poor agreement for most of the data points when for example a multiple of 1 or 2 or 4 or 5 was used. The model uses a value of $C_n = 2.7 \times 10^7$ obtained by calibrating the yield [14,16,35], at an experimental point of 0.5 MA.

The thermonuclear component is also computed in every case and it is found that this component is negligible when compared with the beam-target component. It might be argued that an adjustment to the thermonuclear component could also be attempted in a similar way to the usage of the multiple to $V_{\text{max}}$. However, the usage of the multiple to $V_{\text{max}}$ has some experimental basis due to ion energy measurements. Moreover the value of $V_{\text{max}}$ in each numerical experiment is calculated from the slug model leading to the slow compression phase, whilst it is known experimentally that after the slow compression phase, instability effects set in which will increase the electric fields operating within the pinch. These are the basic arguments supporting the view that the operational beam energy has a value above $V_{\text{max}}$. For the thermonuclear component a feasible model to adjust the yield upwards has yet to be suggested.
7.2. Scaling laws for neutrons from numerical experiments over a range of energies from 10 kJ to 25 MJ

We apply the Lee model code to the MJ machine PF1000 over a range of $C_0$ to study the neutrons emitted by PF1000-like bank energies from 10 kJ to 25 MJ. As shown earlier the PF1000 current trace has been used to fit the model parameters, with very good fitting achieved between the computed and measured current traces (Fig 7). Once the model parameters have been fitted to a machine for a given gas, these model parameters may be used with some degree of confidence when operating parameters such as the voltage are varied [35, 39]. With no measured current waveforms available for the higher megajoule numerical experiments, it is reasonable to keep the model parameters that we have got from the PF1000 fitting.

The optimum pressure for this series of numerical experiments is 10 torr and the ratio $c=b/a$ is retained at 1.39. For each $C_0$, anode length $z_0$ is varied to find the optimum. For each $z_0$, anode radius $a_0$ is varied so that the end axial speed is 10 cm/μs. The numerical experiments were carried out for $C_0$ ranging from 14 μF to 39960 μF corresponding to energies from 8.5 kJ to 24.5 MJ [10]. For this series of experiments we find that the $Y_n$ scaling changes from $Y_n \sim E_0^{2.0}$ at tens of kJ to $Y_n \sim E_0^{0.84}$ at the highest energies (up to 25 MJ) investigated in this series. This is shown in Fig 10.

![Figure 10](image.png)

**Figure 10.** $Y_n$ plotted as a function of $E_0$ in log-log scale, showing $Y_n$ scaling changes from $Y_n \sim E_0^{2.0}$ at tens of kJ to $Y_n \sim E_0^{0.84}$ at the highest energies (up to 25 MJ). This scaling deterioration is discussed in Section 9.2

The scaling of $Y_n$ with $I_{\text{peak}}$ and $I_{\text{pinch}}$ over the whole range of energies investigated up to 25 MJ (shown in Figure 11) are as follows:

\[
Y_n = 3.2 \times 10^{11} I_{\text{pinch}}^{4.5} \quad \text{and} \quad Y_n = 1.8 \times 10^{10} I_{\text{peak}}^{3.8}
\]

where $I_{\text{peak}}$ ranges from 0.3 MA to 5.7 MA and $I_{\text{pinch}}$ ranges from 0.2 MA to 2.4 MA.
Figure 11. Log($Y_n$) scaling with Log($I_{\text{peak}}$) and Log($I_{\text{pinch}}$), for the range of energies investigated, up to 25 MJ

This compares to an earlier study carried out on several machines with published current traces and $Y_n$ yield measurements, operating conditions and machine parameters including the Chilean PF400J, the UNU/ICTP PFF, the NX2 and Poseidon providing a slightly higher scaling laws:

$$Y_n \sim I_{\text{pinch}}^{4.7} \quad \text{and} \quad Y_n \sim I_{\text{peak}}^{3.9}$$

The slightly higher value of the scaling is because those machines fitted are of mixed 'c' mixed bank parameters, mixed model parameters and currents generally below 1MA and voltages generally below the 35 kV [35].

7.3. Summary of neutron scaling laws from numerical experiments:

Over wide ranges of energy, optimizing pressure, anode length and radius, the scaling laws for $Y_n$ [10,35,41,43] obtained through numerical experiments are listed here:

$$Y_n=3.2\times10^{11} I_{\text{pinch}}^{4.5}$$

$$Y_n=1.8\times10^{10} I_{\text{peak}}^{3.8} \quad I_{\text{peak}} \ (0.3 \text{ to } 5.7), I_{\text{pinch}} \ (0.2 \text{ to } 2.4) \ \text{in MA.}$$

$$Y_n\sim E_0^{2.0} \ \text{at tens of kJ to}$$

$$Y_n\sim E_0^{0.84} \ \text{at MJ level (up to 25MJ)}$$

These laws provide useful references and facilitate the understanding of present plasma focus machines. More importantly, these scaling laws are also useful for design considerations of new plasma focus machines particularly if they are intended to operate as optimized neutron sources.

8. Insight 3-Scaling Laws for Soft X-ray Yield

8.1. Computation of Neon SXR yield

We note that the transition from Phase 4 to Phase 5 is observed in laboratory measurements to occur in an extremely short time with plasma/current disruptions resulting in localized regions of high densities and temperatures. These localized regions are not modelled in the code, which consequently computes only an average uniform density, and an average uniform temperature
which are considerably lower than measured peak density and temperature. However, because the 4 model parameters are obtained by fitting the computed total current waveform to the measured total current waveform, the model incorporates the energy and mass balances equivalent, at least in the gross sense, to all the processes which are not even specifically modelled. Hence the computed gross features such as speeds and trajectories and integrated soft x-ray yields have been extensively tested in numerical experiments for several machines and are found to be comparable with measured values.

In the code [14,16,44], neon line radiation $Q_L$ is calculated as follows:

$$\frac{dQ_L}{dt} = -4.6 \times 10^{-31} n_i^2 Z^2 n \left( \frac{\pi r_p^2}{n_i} \right) / T$$  \hspace{1cm} (4)$$

where for the temperatures of our interest we take the SXR yield $Y_{sxr} = Q_L$. $Z_n$ is atomic number.

Hence the SXR energy generated within the plasma pinch depends on the properties: number density $n_i$, effective charge number $Z$, pinch radius $r_p$, pinch length $z_f$ and temperature $T$. It also depends on the pinch duration since in our code $Q_L$ is obtained by integrating over the pinch duration.

This generated energy is then reduced by the plasma self-absorption which depends primarily on density and temperature; the reduced quantity of energy is then emitted as the SXR yield. These effects are included in the modelling by computing volumetric plasma self-absorption factor $A$ derived from the photonic excitation number $M$ which is a function of $Z_n$, $n_i$, $Z$ and $T$. However, in our range of operation, the numerical experiments show that the self absorption is not significant. It was first pointed out by Liu Mahe [22,25] that a temperature around 300 eV is optimum for SXR production. Shan Bing’s subsequent work [23] and our experience through numerical experiments suggest that around $2 \times 10^6$ K (below 200 eV) or even a little lower could be better. Hence unlike the case of neutron scaling, for SXR scaling there is an optimum small range of temperatures ($T$ windows) to operate.

8.2. Scaling laws for neon SXR over a range of energies from 0.2 kJ to 1 MJ

We next use the Lee model code to carry out a series of numerical experiments over the energy range 0.2 kJ to 1 MJ [44]. In this case we apply it to a proposed modern fast plasma focus machine with optimised values for $c$ the ratio of the outer to inner electrode radius and $L_0$ obtained from our numerical experiments.

The following parameters are kept constant: (i) the ratio $c=b/a$ (kept at 1.5, which is practically optimum according to our preliminary numerical trials; (ii) the operating voltage $V_0$ (kept at 20 kV); (iii) static inductance $L_0$ (kept at 30 nH, which is already low enough to reach the $I_{pinch}$ limitation regime [37,38] over most of the range of $E_0$ we are covering) and; (iv) the ratio of stray resistance to surge impedance $RESF$ (kept at 0.1, representing a higher performance modern capacitor bank). The model parameters [46] $f_{in}$, $f_c$, $f_{mr}$, $f_{sc}$ are also kept at fixed values 0.06, 0.7, 0.16 and 0.7. We choose the model parameters so they represent the average values from the range of machines that we have studied. A typical example of a current trace for these parameters is shown in Fig 12.
The storage energy $E_0$ is varied by changing the capacitance $C_0$. Parameters that are varied are operating pressure $P_0$, anode length $z_0$ and anode radius $a$. Parametric variation at each $E_0$ follows the order; $P_0$, $z_0$ and $a$ until all realistic combinations of $P_0$, $z_0$ and $a$ are investigated. At each $E_0$, the optimum combination of $P_0$, $z_0$ and $a$ is found that produces the biggest $Y_{sxr}$. In other words at each $E_0$, a $P_0$ is fixed, a $z_0$ is chosen and $a$ is varied until the largest $Y_{sxr}$ is found. Then keeping the same values of $E_0$ and $P_0$, another $z_0$ is chosen and $a$ is varied until the largest $Y_{sxr}$ is found. This procedure is repeated until for that $E_0$ and $P_0$, the optimum combination of $z_0$ and $a$ is found. Then keeping the same value of $E_0$, another $P_0$ is selected. The procedure for parametric variation of $z_0$ and $a$ as described above is then carried out for this $E_0$ and new $P_0$ until the optimum combination of $z_0$ and $a$ is found. This procedure is repeated until for a fixed value of $E_0$, the optimum combination of $P_0$, $z_0$ and $a$ is found. The procedure is then repeated with a new value of $E_0$. In this manner after systematically carrying out some 2000 runs, the optimized runs for various energies are tabulated in Table 5. We plot $Y_{sxr}$ against $E_0$ as shown in Fig 13.

Table 5. Optimised configuration found for each $E_0$. Optimisation carried out with RESF = 0.1, $c = 1.5$, $L_0 = 30$ nH and $V_0 = 20$ kV and model parameters $f_m$, $f_c$, $f_{mr}$, $f_{cr}$ are fixed at 0.06, 0.7, 0.16 and 0.7 respectively. The $v_a$, $v_s$ and $v_p$ are the peak axial, radial shock and radial piston speeds respectively.

<table>
<thead>
<tr>
<th>$E_0$ (kJ)</th>
<th>$C_0$ (μF)</th>
<th>$a$ (cm)</th>
<th>$z_0$ (cm)</th>
<th>$P_0$ (Torr)</th>
<th>$I_{peak}$ (kA)</th>
<th>$I_{pinch}$ (kA)</th>
<th>$v_a$ (cm/μs)</th>
<th>$v_s$ (cm/μs)</th>
<th>$v_p$ (cm/μs)</th>
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</thead>
<tbody>
<tr>
<td>0.2</td>
<td>1</td>
<td>0.58</td>
<td>0.5</td>
<td>4.0</td>
<td>100</td>
<td>68</td>
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Figure 13. Ysxr vs E0. The parameters kept constants are: RESF=0.1, c=1.5, L0=30nH and V0=20 kV and model parameters fm, fc, fmr, fcr at 0.06, 0.7, 0.16 and 0.7 respectively. The scaling deterioration observed in this Figure is similar to that for neutron yield and is discussed in section 9.2.

We then plot Ysxr against Ipeak and Ipinch and obtain SXR yield scales as

\[ Y_{sxr} \sim I_{\text{pinch}}^{3.6} \]

and

\[ Y_{sxr} \sim I_{\text{peak}}^{3.2}. \]

The Ipinch scaling has less scatter than the Ipeak scaling. We next subject the scaling to further test when the fixed parameters RESF, c, L0 and V0 and model parameters fm, fc, fmr, fcr are varied. We add in the results of some numerical experiments using the parameters of several existing plasma focus devices including the UNU/ICTP PFF (RESF = 0.2, c = 3.4, L0 = 110 nH and V0 = 14 kV with fitted model parameters fm = 0.05, fc = 0.7, fmr = 0.2, fcr = 0.8) [14,16,22,23,25,32,48], the NX2 (RESF = 0.1, c = 2.2, L0 = 20 nH and V0 = 11 kV with fitted model parameters fm = 0.10, fc = 0.7, fmr = 0.12, fcr = 0.68) [14,16,22,23,31,46], and PF1000 (RESF = 0.1, c = 1.39, L0 = 33 nH and V0 = 27 kV with fitted model parameters fm = 0.1, fc = 0.7, fmr = 0.15, fcr = 0.7) [14,16,36,37]. These new data points (white data points in Fig. 14) contain wide ranges of c, V0, L0 and model parameters. The resulting Ysxr versus Ipinch log-log curve remains a straight line, with the scaling index 3.6 unchanged and with no more scatter than before. However the resulting Ysxr versus Ipeak curve now exhibits considerably larger scatter and the scaling index has changed slightly (note the change is not shown/obvious here).

Figure 14. Ysxr is plotted as a function of I_{\text{pinch}} and I_{\text{peak}}. The parameters kept constant for the black data points are: RESF = 0.1, c = 1.5, L0 = 30nH and V0 = 20 kV and model parameters fm, fc, fmr, fcr at 0.06, 0.7, 0.16 and 0.7 respectively. The white data points are for specific machines which have different values for the parameters c, L0 and V0.
We would like to highlight that the consistent behaviour of $I_{\text{pinch}}$ in maintaining the scaling of $Y_{\text{sxr}} \sim I_{\text{pinch}}^{3.6}$ with less scatter than the $Y_{\text{sxr}} \sim I_{\text{peak}}^{3.2}$ scaling particularly when mixed-parameters cases are included, strongly supports the conclusion that $I_{\text{pinch}}$ scaling is the more universal and robust one. Similarly conclusions on the importance of $I_{\text{pinch}}$ in plasma focus performance and scaling laws have been reported [36].

It may also be worthy of note that our comprehensively surveyed numerical experiments for Mather configurations in the range of energies 0.2 kJ to 1 MJ produce an $I_{\text{pinch}}$ scaling rule for $Y_{\text{sxr}}$ not compatible with Gates’ rule [53]. However it is remarkable that our $I_{\text{pinch}}$ Scaling index of 3.6, obtained through a set of comprehensive numerical experiments over a range of 0.2 kJ to 1 MJ, on Mather-type devices, is within the range of 3.5 to 4 postulated on the basis of sparse experimental data, (basically just two machines one at 5 kJ and the other at 0.9 MJ), by Filippov [54], for Filippov configurations in the range of energies 5 kJ to 1 MJ.

It must be pointed out that the results represent scaling for comparison with baseline plasma focus devices that have been optimized in terms of electrode dimensions. It must also be emphasized that the scaling with $I_{\text{pinch}}$ works well even when there are some variations in the actual device from $L_0 = 30$ nH, $V_0 = 20$ kV and $c = 1.5$.

8.3. Summary of Soft X-ray scaling laws found by numerical experiments:

Over wide ranges of energy, optimizing pressure, anode length and radius, the scaling laws for neon SXR are found by numerical experiments to be:

\[ Y_{\text{sxr}} = 8.3 \times 10^{3} \times I_{\text{pinch}}^{3.6} \]

\[ Y_{\text{sxr}} = 600 \times I_{\text{peak}}^{3.2} \; ; \; I_{\text{peak}} \; (0.1 \; \text{to} \; 2.4), \; I_{\text{pinch}} \; (0.07 \; \text{to} \; 1.3) \; \text{in MA}. \]

\[ Y_{\text{sxr}} \sim E_0^{1.6} \; \text{(kJ range)} \]

\[ Y_{\text{sxr}} \sim E_0^{0.8} \; \text{(towards MJ)}. \]

These laws provide useful references and facilitate the understanding of present plasma focus machines. More importantly, these scaling laws are also useful for design considerations of new plasma focus machines particularly if they are intended to operate as neon SXR sources.

9. Insight 4- Neutron Saturation

Besides being accurately descriptive and related to wide-ranging experimental reality, desirable characteristics of a model include predictive and extrapolative scaling. Moreover a useful model should be accessible, usable and user-friendly and should be capable of providing insights. Insight however cannot be a characteristic of the model in isolation, but is the interactive result of the model with the modeler or model user.

It was observed early in plasma focus research [5,9] that neutron yield $Y_n \sim E_0^2$ where $E_0$ is the capacitor storage energy. Such scaling gave hopes of possible development as a fusion energy source. Devices were scaled up to higher $E_0$. It was then observed that the scaling deteriorated, with $Y_n$ not increasing as much as suggested by the $E_0^2$ scaling. In fact some experiments were interpreted as evidence of a neutron saturation effect [5] as $E_0$ approached several hundreds of kJ. As recently as 2006 Krauz [55] and November 2007, Scholz [56] have questioned whether the neutron saturation was due to a fundamental cause or to avoidable machine effects such as incorrect formation of plasma current sheath arising from impurities or sheath instabilities. We
should note here that the region of discussion (several hundreds of kJ approaching the MJ region) is in contrast to the much higher energy region discussed by Schmidt at which there might be expected to be a decrease in the role of beam target fusion processes[5].

9.1. The global neutron scaling law

Recent extensive numerical experiments [10,11,35,42,51] also showed that whereas at energies up to tens of kJ the \( Y_n \sim E_0^{2} \) scaling held, deterioration of this scaling became apparent above the low hundreds of kJ. This deteriorating trend worsened and tended towards \( Y_n \sim E_0^{0.8} \) at tens of MJ. The results of these numerical experiments are summarized in Fig.1 (Section 2 above) with the solid line representing results from numerical experiments. Experimental results from 0.4 kJ to MJ, compiled from several available published sources [4,5,9,13,15,35, 55-58], are also included as squares in the same figure. The combined experimental and numerical experimental results [11,42,51] (see Fig 1 in Section 2 above) appear to have general agreement particularly with regards to the \( Y_n \sim E_0^{2} \) at energies up to 100 kJ, and the deterioration of the scaling from low hundreds of kJ to the 1 MJ level. The global data of Fig. 1 suggests that the apparently observed neutron saturation effect is overall not in significant variance with the deterioration of the scaling shown by the numerical experiments.

9.2. The dynamic resistance

A simple yet compelling analysis of the cause of this neutron saturation has been published [11]. In Fig. 3 (see Section 4 above) on the left side is shown a schematic of the plasma dynamics in the axial phase of the Mather-type plasma focus. In that work the simplest representation was used, in which the current sheet is shown to go from the anode to the cathode perpendicularly. Observation shows that there is actually a canting of the current sheet [18,26,50] and also that only a fraction (typically 0.7) of the total current participates in driving the current sheet. These points are accounted for in the modelling by model parameters \( f_m \) and \( f_c \). We now represent the plasma focus circuit in Fig 15.

![Figure 15. Plasma focus circuit schematic. The capacitor bank with static inductance \( L_0 \) and stray resistance \( r_0 \) is switched into the plasma focus tube where a fraction \( f_c \) of the circuit current \( I(t) \) effectively drives the plasma creating a time-varying inductance \( L(t) \) in the focus tube.

We consider only the axial phase. By surveying published results of all Mather-type experiments we find that all deuterium plasma focus devices operate at practically the same speeds [6] and are characterized by a constancy of energy density (per unit mass) over the whole range of devices from the smallest sub-kJ to the largest MJ devices. The time varying tube inductance is
\( L = \left( \mu / 2 \pi \right) \ln(c) z \), where \( c = b/a \) and \( \mu \) is the permeability of free space. The rate of change of inductance is \( dL/dt = 2 \times 10^{-7} \text{Hz} \) \( dz/dt \) in SI units. Typically on switching, as the capacitor discharges, the current rises towards its peak value, the current sheet is accelerated, quickly reaching nearly its peak speed and continues accelerating slightly towards its peak speed at the end of the axial phase. Thus for most of its axial distance the current sheet is travelling at a speed close to the end-axial speed. In deuterium the end-axial speed is observed to be about 10 cm/\( \mu \)s over the whole range of devices [6]. This fixes the rate of change of inductance \( dL/dt \) as \( 1.4 \times 10^{-2} \) H/s for all the devices, if we take the radius ratio \( c = b/a = 2 \). This value of \( dL/dt \) changes by at most a factor of 2, taking into account the variation of \( c \) from low values of 1.4 (generally for larger machines) to 4 (generally for smaller machines). This typical \( dL/dt \) may also be expressed as 14 m\( \Omega \).

We need now to inquire into the nature of the change in the inductance \( L(t) \). Consider instantaneous power \( P \) delivered to \( L(t) \) by a change in \( L(t) \)

\[
\text{Induced voltage: } V = d(LI)/dt = I(dL/dt) + LI(dI/dt)
\]

Hence instantaneous power into \( L(t) \): \( P = VI = I^2(dL/dt) + LI(dI/dt) \)

Next, consider instantaneous power associated with the inductive energy (\( \%L^2 \)):

\[
P_L = d(\%L^2)/dt = \%L^2(dL/dt) + LI(dI/dt)
\]

We note that \( P_L \) of Eq (7) is not the same as \( P \) of Eq (6). The difference= \( P_L - P \) is not associated with the inductive energy stored in \( L \). We conclude that whenever \( L(t) \) changes with time, the instantaneous power delivered to \( L(t) \) has a component that is not inductive. Hence this component of power \( \%L^2(dL/dt) \) must be resistive in nature; and the quantity \( \%L^2(dL/dt) \) also denoted as half Ldot is identified as a resistance, due to the motion associated with \( dL/dt \); which we call the dynamic resistance \( DR \) [10,11,39,42,51]. Note that this is a general result and is independent of the actual processes involved. In the case of the plasma focus axial phase, the motion of the current sheet imparts power to the shock wave structure with consequential shock heating, Joule heating, ionization, radiation etc. The total power imparted at any instant is just the amount \( \%L^2(dL/dt) \), with this amount powering all consequential processes. We denote the dynamic resistance of the axial phase as \( DR_0 \).

We have thus identified for the axial phase of the plasma focus a typical dynamic resistance of 7 m\( \Omega \) due to the motion of the current sheet at 10 cm/\( \mu \)s. It should be noted here that similar ideas of the role of \( dL/dt \) as a resistance were discussed by Bernard et al [5]. In that work the effect of \( dL/dt \) was discussed only for the radial phase. In our opinion the more important phase for the purpose of neutron saturation is actually the axial phase for the Mather-type plasma focus.

9.3. The interaction of a constant dynamic resistance with a reducing generator impedance causes deterioration in current scaling

We now resolve the problem into its most basic form as follows. We have a generator (the capacitor charged to 30 kV), with an impedance of \( Z_0 = (L_0/C_0)^{0.5} \) driving a load with a near constant resistance of 7 m\( \Omega \). We also assign a value for stray resistance of 0.1\( Z_0 \). This situation may be shown in Table 6 where \( L_0 \) is given a typical value of 30 nH. We also include in the last column the results from a circuit (L-C-R) computation, discharging the capacitor with initial
voltage 30 kV into a fixed resistance load of 7 mΩ simulating the effect of the \(DR_0\) and a stray resistance of value \(0.1Z_0\).

**Table 6.** Discharge characteristics of equivalent PF circuit, illustrating the ‘saturation’ of \(I_{\text{peak}}\) with increase of \(E_0\) to very large values. The last column presents results using circuit (L-C-R) computation, with a fixed resistance load of 7 mΩ, simulating the effect of the \(DR_0\) and a stray resistance of value \(0.1Z_0\).

<table>
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<tr>
<th>(E_0) (kJ)</th>
<th>(C_0) ((\mu)F)</th>
<th>(Z_0) (mΩ)</th>
<th>(DR_0) (mΩ)</th>
<th>(Z_{\text{total}}) (mΩ)</th>
<th>(I_{\text{peak}}=V_0/Z_{\text{total}}) (kA)</th>
<th>(I_{\text{peak},\text{L-C-R}}) (kA)</th>
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Plotting the peak current as a function of \(E_0\) we obtain Fig 16, which shows the tendency of the peak current towards saturation as \(E_0\) reaches large values; the deterioration of the curve becoming apparent at the several hundred kJ level. This is the case for \(I_{\text{peak}}=V_0/Z_{\text{total}}\) and also for the L-C-R discharge with simulated value of the \(DR_0\). In both cases it is seen clearly that a capacitor bank of voltage \(V_0\) discharging into a constant resistance such as \(DR_0\) will have a peak current \(I_{\text{peak}}\) approaching an asymptotic value of \(I_{\text{peak}}=V_0/DR_0\) when the bank capacitance \(C_0\) is increased to such large values that the value of \(Z_0=(L_0/C_0)^{0.5}<<DR_0\). Thus \(DR_0\) causes current ‘saturation’.

**Figure 16.** \(I_{\text{peak}}\) vs \(E_0\) on log-log scale, illustrating \(I_{\text{peak}}\) ‘saturation’ at large \(E_0\)
9.4. Deterioration in current scaling causes deterioration in neutron scaling

In Section 7.2 we had shown the following relationships between $Y_n$ and $I_{\text{peak}}$ and $I_{\text{pinch}}$ as follows:

\[ Y_n \approx I_{\text{pinch}}^{4.5} \]  
\[ Y_n \approx I_{\text{peak}}^{3.8} \]  

(8)  

(9)

Hence saturation of $I_{\text{peak}}$ will lead to saturation of $Y_n$.

At this point we note that if we consider that only 0.7 of the total current takes part in driving the current sheet, as typically agreed upon from experimental observations, then there is a correction factor which reduces the axial dynamic resistance by some 40%. That would raise the asymptotic value of the current by some 40%; nevertheless there would still be ‘saturation’.

Thus we have shown that current ‘saturation’ is inevitable as $E_0$ is increased to very large values by an increase in $C_0$, simply due to the dominance of the axial phase dynamic resistance. This makes the total circuit impedance tend towards an asymptotic value which approaches the dynamic resistance at infinite values of $E_0$. The ‘saturation’ of current inevitably leads to a ‘saturation’ of neutron yield. Thus the apparently observed neutron ‘saturation’ which is more accurately represented as a neutron scaling deterioration is inevitable because of the dynamic resistance. In line with current plasma focus terminology we will continue to refer to this scaling deterioration as ‘saturation’. The above analysis applies to the Mather-type plasma focus. The Filippov-type plasma focus does not have a clearly defined axial phase. Instead it has a lift-off phase and an extended pre-pinch radial phase which determine the value of $I_{\text{peak}}$. During these phases the inductance of the Filippov discharge is changing, and the changing $L(t)$ will develop a dynamic resistance which will also have the same current ‘saturation’ effect as the Filippov bank capacitance becomes big enough.

The same scaling deterioration is also observed in the yield of Neon SXR (see Figure 13) and we expect for other radiation yields as well. The speed restrictions for a plasma focus operating in neon is not the same as that in deuterium. Nevertheless there is a speed window related to the optimum temperature window. This again requires fixing the dynamic resistance of the axial phase for the neon plasma focus within certain limits typically the dynamic resistance equivalent to an axial speed range of 5-8 cm per microsecond. This dynamic resistance and its interaction with the capacitor bank impedance as storage energy is increased is again the cause of the scaling deterioration.

9.5. Beyond presently observed neutron saturation regimes

Moreover the ‘saturation’ as observed in presently available data is due also to the fact that all tabulated machines operate in a narrow range of voltages of 15-50 kV. Only the SPEED machines, most notably SPEED II [59] operated at low hundreds of kV. No extensive data have been published from the SPEED machines. Moreover SPEED II, using Marx technology, has a large bank surge impedance of 50 m$\Omega$ which itself would limit the current. If we operate a range of such high voltage machines at a fixed high voltage, say 300 kV, with ever larger $E_0$ until the surge impedance becomes negligible due to the very large value of $C_0$, then the ‘saturation’ effect would still be there, but the level of ‘saturation’ would be proportional to the voltage. In this way we can go far above presently observed levels of neutron ‘saturation’; moving the research, as it were into presently beyond-saturation regimes.
Could the technology be extended to 1MV? That would raise $I_{\text{peak}}$ to beyond 15 MA and $I_{\text{pinch}}$ to over 6 MA. Also multiple Blumleins at 1 MV, in parallel, could provide driver impedance of 100 mΩ, matching the radial phase dynamic resistance and provide fast rise currents peaking at 10 MA with $I_{\text{pinch}}$ value of perhaps 5 MA. Bank energy would be several MJ. The push to higher currents may be combined with proven neutron yield enhancing methods such as doping deuterium with low % of krypton [60]. Further increase in pinch current might be by fast current injection near the start of the radial phase. This could be achieved with charged particle beams or by circuit manipulation such as current-stepping [11,61,62]. The Lee model is ideally suited for testing circuit manipulation schemes.

10. Neutron Scaling- Its relationship with the plasma focus properties

In Section 2 we had discussed the global scaling law for neutron yield as shown in Figure 1 which was compiled with data from experiments and numerical experiments. Figure 1 shows that whereas at energies up to tens of kJ the $Y_n \sim E_0^2$ scaling held, deterioration of this scaling became apparent above the low hundreds of kJ. This deteriorating trend worsened and tended towards $Y_n \sim E_0^{0.8}$ at tens of MJ. The global data of Figure 1 suggests that the apparently observed neutron saturation effect is overall not in significant variance with the deterioration of the scaling shown by the numerical experiments.

10.1. Relationship with plasma focus scaling properties

Now we link up this neutron scaling law deterioration and subsequent saturation with the scaling properties of the plasma focus discussed in Section 3. This scaling law deterioration and saturation is due to the constancy of the speed factor $SF$ and energy density, as $E_0$ increases. The constancy of the axial speed or $SF$ caused the deterioration of current scaling, requiring that the anode radius ‘a’ is not increased as much as it would have been increased if there were no deterioration. This implies that the size and duration of the focus pinch are also restricted by the scaling deterioration. Ultimately at high tens of MJ, $I_{\text{peak}}$ saturates, the anode radius of the focus should not be increased anymore with $E_0$. The size and duration of the focus pinch no longer increase with $E_0$ and $Y_n$ also saturates. We now have the complete picture.

We may consider the other effects such as the current limitation effect as inductance is reduced and the scaling laws of plasma focus for SXR yield. These are all related to the behaviour of the scaling properties and the interaction of these scaling properties, particularly the dynamic resistance with the capacitor bank impedance.

11. New Development-The 6-phase model- Instability phase fitted by anomalous resistance(s)

11.1. Low inductance plasma focus Type T1- Computed current trace well fitted to measured current trace using 5-phase model

The Lee code does not model the transition from Phase 4 to Phase 5. Nevertheless it has been found to be adequate for modelling all the well-known plasma focus with low static inductance $L_0$ [14-16,32,35,47,49] which we have fitted; in the sense that the computed current traces can be fitted to the measured current trace by adjustment of the model parameters $f_m$, $f_s$, $f_{mr}$ and $f_{cr}$. This has been the case for the PF1000, PF400J, NX1, NX2, DPF78, Poseidon [14], FMPF1 [63], FN-II [49]. Some examples are shown in figures 17.
11.2. High inductance plasma focus Type T2- Computed current trace cannot be fitted to measured current trace using 5-phase model

Amongst the well-published plasma focus devices only the UNU/ICTP PFF [14,16,25-27] which has relatively higher $L_0$ of 110 nH presented less certainty in the fitting. This was due to a very small computed current dip and a measured current dip that has always been masked by very large oscillations taken to be noise; although when operated in unusually low pressure regime, a clear discrepancy was noted between the computed and measured current trace [64].

Recently a current trace from the newly commissioned KSU DPF (Kansas State University Dense Plasma Focus) [65] which has an even higher $L_0$, was obtained by numerically integrating the output of a $dI/dt$ coil. An analysis of the frequency response of the coil system and the DSO signal acquisition system showed that noise frequencies below 200 MHz were removed by the numerical integration. The resultant waveform is clean and clearly shows an extended current dip with good depth and duration (see figure 18, the darker trace).
Figure 18. Computed current trace (lighter trace) with best attempt to fit to the measured current trace (darker trace).

Following the usual procedure of the Lee model code, an attempt was made to fit the computed current trace with the measured. The computed current trace has only a small dip as is characteristic of the computed current dip of a device with large static inductance $L_0$. All possible adjustments were made to the model parameters but the computed current dip could not be made to fit the whole measured current dip. The best fit is shown in figure 18; which shows that the computed dip does fit the first small part of the measured current dip. But the measured dip continues on in both depth and duration far beyond the computed dip.

11.3. Factors distinguishing the two types of plasma focus devices

The code models the electrodynamic situation using the slug model and a reflected shock for the radial phase, ending the radial phase in phase 4. Let's call the radial phase modeled in that manner as the REGULAR radial phase. This REGULAR radial phase, in increasing sharply the inductance of the system (constituting also a dynamic resistance [10,11]) causes a dip on the current trace. Call this the regular dip $RD$. At the end of the REGULAR radial phase experimental observations point to another phase [5,55,56], which we shall call phase 4a, (i.e. after phase 4, but before the final axial phase, called phase 5), of 'instabilities' manifesting in anomalous resistance. These effects would also extract energy from the magnetic field and hence produce further current dips. These effects are not modeled specifically in the code. Call this the extended current dip $ED$.

However it may be argued that as long as the model parameters can be stretched sufficiently to have the computed current dip agree with the measured current dip, then in a gross sense, the modelling is energetically and mass-wise equivalent to the physical situation. Then the resulting gross characteristics from the model would give a fair representation of the actual plasma properties, even though the model has not specifically modeled $ED$. In other words $RD$ is able to be stretched to also model $ED$, with equivalent energetics and mass implications. Whether $RD$ can be stretched sufficiently to cover $ED$ depends on the relative sizes of the two effects. If $RD$ is already a big dip, then this effect may dominate and it is more likely that $RD$ may be stretched sufficiently to cover the less prominent $ED$. If $RD$ is only a miniscule dip and $ED$ is a big dip, then it is unlikely that the $RD$ can be stretched enough to encompass the $ED$.

We attempt to establish criteria for discriminating the types. Noting that generally a plasma focus with small $L_0$, for example the PF1000 with $L_0=33$ nH, exhibits a large computed $RD$ (see figure 16) whereas a plasma focus with a large $L_0$, for example the KSU PF with $L_0=123$ nH, exhibits a small computed $RD$ (see figure 18) we suspect that it has something to do with the inductance $L_0$. 

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or the ratio of $L_0$ with various inductances inherent in the system. We carried out several series of numerical experiments with various configurations similar to existing plasma focus devices, varying the value of $L_0$ in each series and looking at the effect on remnant energies at the end of the RD. Some interesting conclusions may be drawn from a tabulation, such as in Table 7.

<table>
<thead>
<tr>
<th>PF name</th>
<th>$L_0$ (nH)</th>
<th>$C_0$ (μF)</th>
<th>$I_{peak}$ (kA)</th>
<th>$R_L$</th>
<th>$R_{EL}$</th>
<th>RD dip (%)</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poseidon</td>
<td>17.7</td>
<td>156</td>
<td>3205</td>
<td>0.9</td>
<td>2.5</td>
<td>32</td>
<td>T1</td>
</tr>
<tr>
<td>PF1000</td>
<td>33.5</td>
<td>1332</td>
<td>1845</td>
<td>1</td>
<td>1.6</td>
<td>34</td>
<td>T1</td>
</tr>
<tr>
<td>DPF78</td>
<td>55</td>
<td>17.2</td>
<td>869</td>
<td>4.1</td>
<td>12.8</td>
<td>11</td>
<td>T1</td>
</tr>
<tr>
<td>FN-II</td>
<td>75</td>
<td>7.5</td>
<td>309</td>
<td>4.3</td>
<td>8.5</td>
<td>10</td>
<td>T1</td>
</tr>
<tr>
<td>FMPF1</td>
<td>31</td>
<td>2.4</td>
<td>81</td>
<td>6.9</td>
<td>8.6</td>
<td>14.5</td>
<td>T1</td>
</tr>
<tr>
<td>PF-400J</td>
<td>40</td>
<td>1</td>
<td>126</td>
<td>8.8</td>
<td>17.3</td>
<td>8</td>
<td>T1</td>
</tr>
<tr>
<td>UNUICTP</td>
<td>110</td>
<td>30</td>
<td>163</td>
<td>16.7</td>
<td>29.5</td>
<td>1.9</td>
<td>T2</td>
</tr>
<tr>
<td>KSU</td>
<td>123</td>
<td>12.5</td>
<td>137</td>
<td>21.4</td>
<td>40</td>
<td>1.5</td>
<td>T2</td>
</tr>
</tbody>
</table>

We considered the inductance ratio $R_L = (L_0 + L_a)/L_{pinch}$ where $L_{pinch}$ is the inductance of the focus pinch at the end of the REGULAR radial phase, $L_0$ the bank static inductance and $L_a$ the inductance of the axial part of the focus tube. We also considered the remnant energy ratio $R_{EL} = (E_{L0} + E_{La})/E_{Lpinch}$ where $E_{L0}$ = energy stored in $L_0$ at end of the RD, $E_{La}$ = energy stored in $L_a$ at end of the RD and $E_{Lpinch}$ = energy stored inductively in the pinch at end of RD.

Computing the values of these two quantities for PF1000, Poseidon, DPF78, NX2, PF400J, FMPF-1, FNII and UNU/ICTPFF and KSU PF, we have a range of devices from very big (MJ) to rather small (sub kJ) of which we have well documented fittings. These are shown in Table 7 for operation in D$_2$. For other gases there are not many readily available examples. We are able to compile Table 8 for operation in Ne.

<table>
<thead>
<tr>
<th>PF name</th>
<th>$L_0$ (nH)</th>
<th>$C_0$ (μF)</th>
<th>$I_{peak}$ (kA)</th>
<th>$R_L$</th>
<th>$R_{EL}$</th>
<th>RD dip (%)</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>NX2</td>
<td>Ne</td>
<td>20</td>
<td>28</td>
<td>322</td>
<td>1.5</td>
<td>2.6</td>
<td>19</td>
</tr>
<tr>
<td>UNUICTP</td>
<td>Ne</td>
<td>110</td>
<td>30</td>
<td>178</td>
<td>15</td>
<td>26</td>
<td>2.5</td>
</tr>
</tbody>
</table>

Generally we see the trend that the smaller is the ratio $R_L$, the bigger is the regular current dip (RD). When this ratio is large (primarily due to a large $L_0$ in the numerator), like in the case of KSU PF, the REGULAR radial phase RD is miniscule. Likewise, the trend is also observed for the ratio $R_{EL}$. The smaller this energy ratio, the bigger is the current dip. On the basis of these two ratios we have divided the plasma focus devices in Tables 7 and 8 into two types: T1 and T2. Type T1 are for plasma focus devices with relatively small $L_0$ with large RD’s and with relatively small ratios $R_L$ and $R_{EL}.$ These T1 focus devices are well-fitted using the Lee model code. The computed current traces (with radial phase computed only as a regular dip RD) are well-fitted to the whole measured current trace. Type T2 are for plasma focus devices with relatively large $L_0$ with small RD’s and with relatively large ratios of $R_L$ and $R_{EL}.$ These T2 focus devices are not well-fitted using the Lee model code. The computed current trace shows only a small dip which is fitted to the first portion of the measured current dip; but the measured current dip has an extended portion which is not well-fitted using the 5-phase Lee model code.

Next we note that the magnetic energy density per unit mass at the start of the radial phase is the same across the whole range of devices [6]. Thus T1 with a big RD drops the current a lot and
strongly depletes the magnetic energy per unit mass at the end of the RD, leading to a small ED. Consequently T1 are completely fitted using a model that computes only the RD, stretching the model parameters until the large RD covers also the small ED. Conversely a T2 plasma focus has a small RD, consequently a large ED and cannot be completely fitted with the computed RD. Thus a big RD drops the current a lot and strongly depletes the magnetic energy per unit mass at the end of the REGULAR radial phase. Hence a device with small \( R_L \) produces a big RD and ends up with relatively less energy per unit mass at the end of the REGULAR phase when compared to a device with a big value of \( R_L \). Therefore a big RD generally tends to lead to a small ED; whereas a small RD is more conducive to lead to a larger ED.

From the above we may surmised that T1 plasma focus has a big RD, consequently a small ED and hence can be completely fitted using a model that computes only the RD, which is able to stretch its RD by stretching the model parameters until the large RD covers also the small ED. Moreover energetically and mass-wise the fitting is correct. On the other hand T2 plasma focus has a small RD, consequently a large ED. T2 plasma focus cannot be completely fitted with the RD computed from the code, no matter how the model parameters are stretched. To fit the computed current trace to the measured current for T2, a phase 4a needs to be included into the model in order to progress the current dip beyond the small RD into the large ED part of the current dip.

11.4. The anomalus resistance term in the 6-phase Lee Model

It is generally accepted [5] that after the regular dynamic phases ending in the formation of the plasma focus pinch, at the end of the pinch the system becomes unstable, develops a high ‘anomalous’ resistivity and breaks up. The overall processes to start this instability takes an exceedingly short time, the experimental observations indicate that the breakup time is far shorter than the ‘regular’ radial phases. This is evident for example in streak photographs which show that the break up time is less than the duration of the pinch. For example for a small focus of around 10 kJ the breakup time has been measured [66] as 30ns after a duration of some 80 ns for the radial inward shock and reflected shock phases. There appears to be large number of competing instability processes [5], among which are some with exceedingly short time scales. Hence it appears reasonable to assume that the speed at which the plasma can convert the remnant inductive energy into anomalously resistive energy is ultimately limited by the time scales of the gross electrical components which have to supply the energy for the break-up processes.

From a careful study of measured waveforms of current and voltages, various sources have reported that the plasma anomalous resistive voltages are consistent with an ‘anomalous’ resistance of the order of 1 \( \Omega \) [5,66]. Hence the (1/e) time scale (which is \( L/R \)) of current is estimated as 10 ns per 10 nH of inductance \( (L_0+L_a+L_{pinch}) \); that is the time it takes the current to drop to some 36% limited by the lumped components of the circuit. Because inductive energy is proportional to \( I^2 \), this is also the time it would take for the inductive energy to drop to some 14%. On the other hand, the time it takes for inductive energy to drop to 55% (exp[-0.3])\(^2\) is some 3 ns per 10 nH. For a low \( L_0 \) (20-30 nH) plasma focus with this quantity of inductance \( (L_0+L_a+L_{pinch}) \) of around 40 nH this range of time (for inductive energy drop to 14% to 55%) is of the order of 12-40 ns. On the other hand for a high \( L_0 \) plasma focus with \( L_0=120 \) nH and the total inductance being 130 nH, this 14% to 55% inductive energy drop range could be some 40-130 ns. Thus for a low \( L_0 \) system assuming a range of inductive energy drop down to 14 to 55 %, we may estimate a relatively small ED region with small depth and timescale of the order of 12 to 40ns. Whereas for a high \( L_0 \) system we expect a relatively high dip ED region with time scales of the order of 40-130 ns.
Therefore a low $L_0$ system would have a small (in depth and in time) $ED$ which can easily be merged into the larger (in depth and in time) $RD$; the whole current dip being capable of being treated as just the $RD$. On the other hand the high $L_0$ system would have an $ED$ which is large (in both depth and time) when compared with the $RD$; hence the $ED$ has to be separately treated by modelling a phase 4a.

One way to simulate the current $ED$ is to assign the phase 4a period with an anomalous resistance term such as:

$$R = R_0 \left[ \exp(-t/t_2) - \exp(-t/t_1) \right]$$

where $R_0$ is of the order of $1\Omega$, $t_1$ is a characteristic time representative of the rise time of the anomalous resistance and $t_2$ is characteristic of the fall time of the anomalous resistance (Figure 19).

![Figure 19. Simulating anomalous resistance.](image)

### 11.5. The first result of the 6-phase model

We have applied this technique to the KSU current waveform (Figure 18). We note that the computed $RD$ only agrees with a small part of the measured current dip and does not follow the measured current dip which goes on to an $ED$. Following that first current dip in this particular case the dip continues in a second portion which is almost flat then followed by a third section which is less steep than the first dip but of slightly longer duration. We applied a resistance term to each of the 3 sections. We adjusted the parameters $R_0$, $t_2$ and $t_1$ for each of the section as well as a fraction $\text{endfraction}$ which terminates the term. The fitted parameters are as follows:

<table>
<thead>
<tr>
<th>Dip</th>
<th>$R_0$ ($\Omega$)</th>
<th>$T_2$ (ns)</th>
<th>$T_1$ (ns)</th>
<th>$\text{endfraction}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0</td>
<td>70</td>
<td>15</td>
<td>0.53</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
<td>70</td>
<td>40</td>
<td>0.4</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>70</td>
<td>25</td>
<td>1.0</td>
</tr>
</tbody>
</table>

With these parameters it is found that the computed current dip now fits the measured current dip all the way to the end of the current dip. The fitting has involved the fitting of the $RD$, followed by the $ED$ of the first dip; then follow the second and third dips treated as $ED$’s, each requiring a separate anomalous resistance function.
Figure 20. Computed Current (dip region only and expanded to see details) fitted to measured current with inclusion of Phase 4a.

The resistance functions used for the fitting are also shown in Figure 20 (dashed trace, with the resistance values magnified 200 times in order to be visible on the scale of Figure 20). The computed voltage waveform is also shown (trace labeled 2) compared with the measured voltage waveform (trace labeled 1). The correspondence of the computed voltage waveform and the measured is seen clearly [67].

12. Conclusion

This paper has reviewed the extensive and systematic numerical experiments which have been used to uncover new insights into plasma focus fusion devices including the following. A plasma current limitation effect was unexpectedly found, as the static inductance of any focus device is reduced towards very small values. Scaling laws of neutron yield and soft x-ray yield as functions of storage energies, circuit peak current as well as plasma pinch current, were developed over wider range of parameters than attempted previously. This paper has reviewed the global scaling law for neutron yield as a function of storage energy. First, the scaling deterioration and eventual ‘saturation’ of circuit current are ascribed to the energy density constancy manifested in the form of a constancy in dynamic resistance of the axial phase. Second, the deterioration of current scaling implies that the anode radius ‘a’ is not increased as much as it would have been if there were no deterioration. Third, this implies that the size and duration of the focus pinch are also restricted by the scaling deterioration. Ultimately at high tens of MJ, I_{peak} saturates, the anode radius of the focus should not be increased anymore with E_0, the size and duration of the focus pinch no longer increase with E_0.

The restriction on the plasma pinch size and duration has a corresponding effect on the neutron yield Y_n. The neutron yield Y_n scales with E_0^{2} at low energies up to tens of kJ, begins to exhibit scaling deterioration around low hundreds of kJ and approaches ‘saturation’ at high tens of MJ. In this manner this paper has connected the global scaling laws for the current and the neutron yield to the scaling properties of the plasma focus. This more complete picture will facilitate deeper understanding and the further development of the plasma focus as a fusion device. Through these numerical experiments, the cause of neutron ‘saturation’ as device storage energy is increased is found to be the axial phase ‘dynamic resistance’. With the fundamental cause discovered, it is suggested that beyond ‘present saturation’ regimes may be reached by going to higher voltages, and using plasma current enhancement techniques such as current-steps.
Finally a brief discussion is made on the latest development to the Lee Model code, extending it into a 6th phase; the so-called phase 4a which links the end of the pinch phase 4 to the large column axial phase 5. This new development classifies plasma focus into T1 and T2. The T1 devices (with low L₀) have all been fitted well using the 5-phase Lee Model code. The T2 devices are the high inductance (high L₀) type for which the 5-phase model is inadequate. The post-pinch instabilities phase is modeled using an equivalent anomalous resistance which then enables the T2 measured current waveform to be fitted to the measured current waveform in the portion beyond the RD (regular dip) extending the ED (extended dip). The fitting of the anomalous resistance(s) thus results in quantitative experimental data for the instability phase. The anomalous resistance, in effect measured by this method will provide quantitative information for understanding the instabilities.

It is expected that numerical experiments will continue to play a major role complementing and even guiding laboratory measurements and practices. The ground-breaking insights thus gained will completely open up the directions of plasma focus fusion research.

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